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Rotating charged dust in general relativity

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Abstract. The paper considers a distribution of charged dust in rigid motion where the electromagnetic potential vector $A^\mu = kv^\mu$, v^μ being the velocity vector of the dust. Such a relation makes the magnetic field vector everywhere coincident in direction with the vorticity vector. Without introducing any specific symmetry assumption, Maxwell's equations and some of the Einstein equations yield some constraints on the ratio of charge to mass density. The investigation further leads to some coordinate-independent formulae from which it seems possible to construct hitherto unknown solutions of the Einstein–Maxwell equations for distributions of charged dust for both constant and non-constant k.

1. Introduction

In a paper with identical title, Bonnor (1980) has considered the relativistic and classical theories of axially symmetric distributions of rotating charged dust. Specialising further to the case of rigid rotation, Bonnor could obtain the general solution of the Einstein–Maxwell equations in the case of vanishing Lorentz force. However if the Lorentz force did not vanish, Bonnor could obtain only a very special class of solutions with a value two for the ratio between the charge density σ and the mass density ρ. Bonnor wondered about the possible significance of this particular value of the ratio, which nevertheless remained unclarified.

A close examination of these solutions of Bonnor shows that he has effectively assumed in this case the relation $A^\mu = kv^\mu$ between the electromagnetic potential vector A^μ and the velocity vector v^μ of the charged dust with $k^2 = \frac{1}{4}$. The present author felt curious about whether one could proceed by retaining the formal relation but allowing k to have general constant or even non-constant value. Then again we had a hunch that the result of Bonnor regarding the interesting role of the value two for the ratio of charge to mass density may be due essentially to the condition of rigid motion, irrespective of axial symmetry. A reason for such an idea is the theorem previously deduced by De and Raychaudhuri (1968) that for static equilibrium of charged dust, $|\sigma|/\rho$ must be equal to unity irrespective of any symmetry consideration. Could not similar general results hold good for stationary equilibrium as well?

In the present investigation, it is found that assuming rigid rotation, and constancy of k, the general relations given by Raychaudhuri and De (1970) in terms of the electric and magnetic vectors and the characteristics of the velocity field (i.e. the shear, acceleration, vorticity and expansion) yield very simple formulae and one could indeed recover the results obtained by Bonnor although axial symmetry is nowhere assumed. It is also shown that one can have solutions even for $k^2 \neq \frac{1}{4}$, but in that case the Poynting vector in the rest frame of the dust has to vanish. Kinematically this means that the vorticity and acceleration vectors are in the same direction. The investigation has further brought out formulae from which it would perhaps be possible to obtain many hitherto unknown solutions of the Einstein–Maxwell equations with arbitrary values of σ/ρ associated with non-constant values of k.

2. The case of constant k

We introduce the electric and magnetic field vectors as seen by the dust by the defining relations

$$E^{\mu} = F^{\alpha\mu} v_{\alpha}, \tag{2.1}$$

$$B^{\mu} = {}^*F^{\alpha\mu} v_{\alpha} = \tfrac{1}{2}\eta^{\alpha\mu\nu\sigma} F_{\nu\sigma} v_{\alpha}. \tag{2.2}$$

Assuming the already stated relation between the electromagnetic potential and the velocity vector

$$A^{\mu} = k v^{\mu} \tag{2.3}$$

where k in the present section will be assumed to be a constant, by straightforward calculation we now obtain

$$B^{\mu} = -2k\omega^{\mu}, \tag{2.4}$$

$$E^{\mu} = -k\dot{v}^{\mu}, \tag{2.5}$$

where ω^{μ} and $\dot{v}^{\mu}$ are the vorticity and acceleration vectors defined by

$$\omega^{\mu} = \tfrac{1}{2}\eta^{\mu\rho\sigma\nu} v_{\rho;\sigma} v_{\nu}, \tag{2.6}$$

$$\dot{v}^{\mu} = v^{\mu}{}_{;\alpha} v^{\alpha}. \tag{2.7}$$

In the above formulae $\eta^{\alpha\beta\gamma\delta}$ represent the Levi–Civita antisymmetric tensor, the comma and the semicolon stand for ordinary and covariant derivatives respectively.

As we are considering a dust cum electromagnetic field distribution, the Einstein equations are (Lichnerowicz 1967)

$$\begin{aligned} R^{\mu}{}_{\nu} - \tfrac{1}{2}R\delta^{\mu}{}_{\nu} &= -8\pi T^{\mu}{}_{\nu} \\ &= -8\pi[\rho v^{\mu} v_{\nu} - (4\pi)^{-1}(F^{\mu\alpha}F_{\nu\alpha} - \tfrac{1}{4}\delta^{\mu}{}_{\nu}F^{\alpha\beta}F_{\alpha\beta})] \\ &= -8\pi[\rho v^{\mu} v_{\nu} - (4\pi)^{-1}(\tfrac{1}{2}\delta^{\mu}{}_{\nu} - v^{\mu}v_{\nu})(E^2 + B^2) \\ &\quad - (4\pi)^{-1}(E^{\mu}E_{\nu} + B^{\mu}B_{\nu}) - (4\pi)^{-1}(v^{\mu}S_{\nu} + v_{\nu}S^{\mu})] \end{aligned} \tag{2.8}$$

where S^{μ} is the Poynting vector, defined by

$$S^{\mu} = \eta^{\mu\nu\rho\sigma} E_{\nu} B_{\rho} v_{\sigma}, \tag{2.9}$$

and

$$E^2 = -E_{\mu}E^{\mu}, \qquad B^2 = -B_{\mu}B^{\mu}. \tag{2.10}$$

We also have Maxwell's equations

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^{\mu}, \tag{2.11}$$

$${}^*F^{\mu\nu}{}_{;\nu} = 0. \tag{2.12}$$

We assume the current to be purely convectional, so that

$$J^\mu = \sigma v^\mu \tag{2.13}$$

where σ is the charge density. From the divergence of equation (2.8) we obtain

$$\dot{v}^\mu = (\sigma/\rho)E^\mu. \tag{2.14}$$

The Maxwell equations may be reduced to the following set of equations by resolving along and orthogonal to the velocity vector v^μ (Raychaudhuri and De 1970):

$$4\pi\sigma = E^\mu{}_{;\mu} - (\sigma/\rho)E^2 - 2B_\mu\omega^\mu, \tag{2.15}$$

$$2E_\alpha\sigma^{\alpha\beta} - \tfrac{1}{3}\theta E^\beta - v^\alpha(E^\beta{}_{;\alpha} - E_\alpha{}^{;\beta}) + \eta^{\beta\alpha\lambda\sigma}(\dot{v}_\lambda v_\alpha B_\sigma + v_\lambda B_{\sigma;\alpha}) = 0, \tag{2.16}$$

$$B^\mu\dot{v}_\mu + B^\alpha{}_{;\alpha} + 2E_\alpha\omega^\alpha = 0, \tag{2.17}$$

$$2B^\alpha\sigma_\alpha{}^\mu - \tfrac{1}{3}\theta B^\mu - v^\alpha(B^\mu{}_{;\alpha} - B_\alpha{}^{;\mu}) + \eta^{\lambda\alpha\mu\sigma}(\dot{v}_\lambda v_\sigma E_\alpha - v_\lambda E_{\sigma;\alpha}) = 0. \tag{2.18}$$

In the above equations $\sigma_{\alpha\beta}$ and θ are the shear tensor and the expansion scalar respectively, defined as

$$\sigma_{\alpha\beta} = \tfrac{1}{2}(v_{\alpha;\beta} + v_{\beta;\alpha}) - \tfrac{1}{3}(g_{\alpha\beta} - v_\alpha v_\beta)\theta - \tfrac{1}{2}(\dot{v}_\alpha v_\beta + v_\alpha\dot{v}_\beta) \tag{2.19}$$

$$\theta = v^\mu{}_{;\mu}. \tag{2.20}$$

The assumption of rigid motion means that there exists a Killing vector ξ^μ in the direction of the velocity vector v^μ:

$$v^\mu = \lambda\xi^\mu. \tag{2.21}$$

As v^μ is a unit vector,

$$v^\mu v_\mu = 1, \qquad \lambda^2 = (\xi_\mu\xi^\mu)^{-1}.$$

We now have

$$\lambda_{,\mu}\xi^\mu = \sigma_{\alpha\beta} = \theta = 0. \tag{2.22}$$

Using the Killing equation

$$\xi_{(\mu;\nu)} = 0 \tag{2.23}$$

we obtain

$$\dot{v}_\mu = (\log\lambda)_{,\mu}, \tag{2.24}$$

$$E_\mu = (-k\log\lambda)_{,\mu}. \tag{2.25}$$

The Lie derivative of E^μ with respect to ξ^μ vanishes, so that

$$\begin{aligned} 0 &= \xi_{\mu;\alpha}E^\alpha - E_{\mu;\alpha}\xi^\alpha \\ &= -\xi_{\alpha;\mu}E^\alpha - E_{\mu;\alpha}\xi^\alpha \\ &= -(1/\lambda)_{,\mu}v_\alpha E^\alpha - (1/\lambda)v_{\alpha;\mu}E^\alpha - E_{\mu;\alpha}v^\alpha/\lambda. \\ &= (1/\lambda)v^\alpha(E_{\alpha;\mu} - E_{\mu;\alpha}) \end{aligned} \tag{2.26a}$$

where we have used $v^\alpha E_\alpha = 0$. In an exactly similar manner we may show that

$$\nu^\alpha(B_{\alpha;\mu} - B_{\mu;\alpha}) = 0. \tag{2.26b}$$

Using equations (2.22), (2.24) and (2.26*a*), the first three terms in (2.16) vanish and we obtain

$$\eta^{\beta\alpha\lambda\sigma}(\dot{v}_\lambda v_\alpha B_\sigma + v_\lambda B_{\sigma;\alpha}) = 0. \tag{2.27}$$

Equation (2.27) may be written in the form

$$v_\alpha(\dot{v}_\mu B_\sigma - \dot{v}_\sigma B_\mu) + v_\mu(\dot{v}_\sigma B_\alpha - \dot{v}_\alpha B_\sigma) + v_\sigma(\dot{v}_\alpha B_\mu - \dot{v}_\mu B_\alpha)$$
$$+ v_\mu(B_{\sigma;\alpha} - B_{\alpha;\sigma}) + v_\sigma(B_{\alpha;\mu} - B_{\mu;\alpha}) + v_\alpha(B_{\mu;\sigma} - B_{\sigma;\mu}) = 0$$

so that contracting with v^α and using (2.26*b*), we obtain

$$(\dot{v}_\mu B_\sigma - \dot{v}_\sigma B_\mu) + (B_{\mu;\sigma} - B_{\sigma;\mu}) = 0.$$

Substituting from equation (2.24), the above equation gives

$$(B_\sigma/\lambda)_{,\mu} - (B_\mu/\lambda)_{,\sigma} = 0$$

so that we obtain finally

$$B_\mu = \lambda\psi_{,\mu}. \tag{2.28}$$

Using (2.4), (2.5) and (2.28) in equation (2.17), we obtain

$$\psi_{,\mu}{}^{;\mu} = -(3/\lambda)\psi_{,\mu}\lambda^{,\mu}$$

or

$$(\psi^{;\mu}\lambda^3)_{;\mu} = 0. \tag{2.29}$$

In the case when the Poynting vector defined by (2.9) vanishes, the scalar ψ will be a function of λ and (2.28) will assume the form

$$B_\mu = F(\lambda)\lambda_{,\mu}. \tag{2.28b}$$

We shall return to equation (2.28*b*) later.

In view of equations (2.22), (2.5) and (2.25) and taking account of the vanishing of the Lie derivative of B^μ with respect to ξ^μ (cf equation (2.26)), we find that equation (2.18) is trivially satisfied. From equations (2.5) and (2.14) we obtain

$$\sigma/\rho = -1/k. \tag{2.30}$$

Using equations (2.4), (2.30) and (2.15), we have

$$4\pi\sigma = E^\alpha{}_{;\alpha} + (1/k)(E^2 - B^2). \tag{2.31}$$

To proceed further we make use of the identity

$$v^\mu{}_{;\mu;\sigma} - v^\mu{}_{;\sigma;\mu} = R_{\sigma\alpha}v^\alpha. \tag{2.32}$$

The explicit form of the identity (2.32) using equation (2.8) has been worked out by Raychaudhuri and De (1970). We reproduce the results as simplified by taking into consideration equation (2.22):

$$(4\pi\rho + E^2)(1 - \sigma^2/\rho^2) + B^2 = 2(\sigma/\rho)B^\alpha\omega_\alpha + 2\omega^2 + (\sigma/\rho)_{,\mu}E^\mu, \tag{2.33}$$

$$2S^\gamma = \eta^{\mu\nu\beta\gamma}(\omega_{\mu;\beta}v_\nu - 2\omega_\mu v_\nu \dot{v}_\beta). \tag{2.34}$$

In equations (2.33) and (2.34) we have corrected some mistakes regarding signs in the paper of Raychaudhuri and De which we came across in the process of checking.

Substituting from equations (2.4) and (2.30), equation (2.33) becomes, if $k^2 \neq 1$,

$$4\pi\rho + E^2 = -B^2(2k^2+1)/2(k^2-1). \tag{2.35}$$

In the case $k^2 = 1$, equation (2.33) along with (2.30) would give $B^\mu = \omega^\mu = 0$. ξ^μ would then be a hypersurface orthogonal vector and the equilibrium would be called static, and we then recover the result of De and Raychaudhuri (1968). Equation (2.35) shows that $k^2 < 1$, or $|\sigma|/\rho > 1$.

In view of equations (2.27), (2.9) and (2.5) both terms on the right in equation (2.34) may be expressed in terms of the Poynting vector, and doing so we obtain

$$2S^\gamma = S^\gamma/2k^2.$$

so that either

$$S^\gamma = 0 \tag{2.36}$$

or

$$k^2 = \tfrac{1}{4}, \qquad \sigma/\rho = \pm 2. \tag{2.37}$$

Of course there is a third alternative that both (2.36) and (2.37) are valid. Using equation (2.37) in (2.31) and (2.35), we obtain

$$4\pi\rho + E^2 = B^2, \tag{2.38}$$

$$E^\alpha{}_{;\alpha} = 0. \tag{2.39}$$

Equation (2.38) shows that for $\rho > 0$, $B^2 > E^2$, i.e. the magnetic field dominates over the electric field. This result can be made significant if one keeps in mind that the vorticity in our case is proportional to the magnetic intensity, and while both electric and magnetic field energy densities as well as the mass energy density contribute to gravitational 'attraction', equilibrium is due to these being balanced by the 'centrifugal repulsion' and electromagnetic interaction. In view of equation (2.25), equation (2.39) may be written as

$$(\log \lambda)_{,\mu}{}^{;\mu} = 0. \tag{2.40}$$

It is interesting to compare these results with those of Bonnor (1980). While we have pursued the investigation without using any coordinate system or axial symmetry, Bonnor introduced axial symmetry and using a comoving coordinate system could write the line element in the form

$$ds^2 = -e^\mu(dr^2 + dz^2) - l\,d\theta^2 - 2m\,d\theta\,dt + f\,dt^2 \tag{2.41}$$

with

$$fl + m^2 = r^2. \tag{2.42}$$

He then obtained, with the condition (2.37) introduced in a rather *ad hoc* manner, the following equations:

$$\nabla^2(\ln\phi) = 0 \tag{2.43}$$

where

$$\phi = \pm\tfrac{1}{2}f^{1/2}, \tag{2.44}$$

$$\nabla^2 = \partial^2/\partial z^2 + \partial^2/\partial r^2 + (1/r)\partial/\partial r, \tag{2.45}$$

$$r^{-1}(m/f)_{,1}\phi^3 = Y_{,2}, \tag{2.46}$$

$$r^{-1}(m/f)_{,2}\phi^3 = -Y_{,1}, \tag{2.47}$$

$$\nabla^2 Y - 3\phi^{-1}(\phi_{,1}Y_{,1} + \phi_{,2}Y_{,2}) = 0, \tag{2.48}$$

where the subscripts 1 and 2 indicate the coordinates z and r respectively. Comparing the line element (2.41) with equation (2.21) we find, with 4 standing for the time coordinate,

$$\xi^\mu = \delta^\mu{}_4, \tag{2.49}$$

$$\lambda^2 = 1/f, \tag{2.50}$$

$$B_1 = -r^{-1}(m/f)_{,2}\phi^2, \tag{2.51}$$

$$B_2 = r^{-1}(m/f)_{,1}\phi^2. \tag{2.52}$$

Thus equations (2.43), (2.46), (2.47) and (2.48) are identical with our equations (2.40), (2.28) and (2.29) respectively if Bonnor's Y is identified with our ψ. Bonnor summed up his investigation with the words 'a solution is obtained by choosing a harmonic function for ln ϕ and then solving the equation for Y'. In our case where axial symmetry is not assumed, things may not be that simple; nevertheless it is fairly obvious that one can obtain solutions even in more general cases. However, to obtain a complete solution, one would have to introduce a coordinate system with some specific conditions—in the present investigation we propose to confine ourselves to coordinate-independent tensor relations.

It is clear from the foregoing that the specific value two of $|\sigma|/\rho$ need not hold good if S^γ vanishes. We shall now discuss this case. From equations (2.30), (2.31) and (2.35) we obtain, eliminating σ and ρ,

$$-k(\log\lambda)_{,\alpha}{}^{;\alpha} = \frac{4k^2-1}{2k(k^2-1)}B^2.$$

Using now (2.28b) which holds because the Poynting vector vanishes, the above equation becomes

$$q_{,\alpha}{}^{;\alpha} = \frac{4k^2-1}{2k^2(k^2-1)} f^2(q) q_{,\mu} q^{,\mu} \tag{2.53}$$

where

$$q \equiv \log\lambda, \tag{2.54}$$

$$B_\mu = f(q) q_{,\mu}. \tag{2.55}$$

With equation (2.55), equation (2.17) yields in place of (2.29)

$$f(q) q_{,\mu}{}^{;\mu} = -q_{,\mu} q^{,\mu}[2f(q) + f'(q)], \tag{2.56}$$

$f'(q)$ indicating the derivative of $f(q)$ with respect to q. Comparing equations (2.53) and (2.56), we obtain (if $k^2 \neq 1$ or $\frac{1}{4}$)

$$\frac{1}{f^2} + \frac{1}{2}\frac{f'}{f^3} = -\frac{4k^2-1}{4k^2(k^2-1)}$$

which may be readily integrated to give

$$f(q)=\left(Ae^{4q}-\frac{4k^2-1}{4k^2(k^2-1)}\right)^{-1/2} \tag{2.57}$$

where A is an arbitrary constant of integration. Equations (2.53) and (2.57) together can be used to develop complete solutions of the Einstein–Maxwell equations and the situation is especially simple in the case of axial symmetry. However, in keeping with the spirit of the present paper, we desist from doing that and confine ourselves to tensorial relations. The particular case obtained by putting $A = 0$ is somewhat interesting. With f real this is possible only if $\frac{1}{4}<k^2<1$, i.e. if $|\sigma|/\rho$ lies between 2 and 1. In this case equation (2.55) gives

$$\frac{B_\mu}{E_\mu}=\mp 2\left(\frac{1-k^2}{4k^2-1}\right)^{1/2} \qquad (k^2\neq \tfrac{1}{4}).$$

Thus $A = 0$ corresponds to a constant ratio between the magnetic and electric intensities, although the ratio may range from zero to arbitrary large values. If $k^2 = 1$, we have the simple electrostatic case, while for $k^2=\frac{1}{4}$, (2.53) passes over to (2.40), and using equations (2.56) and (2.53) one obtains

$$B_\mu\lambda^3=C\lambda_{,\mu}$$

with C an arbitrary constant. An explicit solution for this case with axial symmetry is given by Bonnor (1980).

In general for $A = 0$, we obtain on substituting from equation (2.57) in (2.53)

$$q_{,\alpha}{}^{;\alpha}=-2q_{,\mu}q^{,\mu}$$

or

$$[\lambda^2]_{,\alpha}{}^{;\alpha}=0. \tag{2.58}$$

This may be compared with equation (2.40) obtained earlier for $S^\gamma\neq 0$.

With $A\neq 0$, a substitution of (2.57) into (2.53) gives after some simplification

$$[\sin^{-1}(a\lambda^2)]_{,\alpha}{}^{;\alpha}=0 \tag{2.59a}$$

or

$$[\sinh^{-1}(a\lambda^2)]_{,\alpha}{}^{;\alpha}=0 \tag{2.59b}$$

according as $4k^2(k^2-1)A/(4k^2-1)$ is positive or negative and

$$a^2=|4k^2(k^2-1)A/(4k^2-1)|.$$

Again the equations (2.59) may be reduced to Laplace's equation in two dimensions in the case of axial symmetry.

3. Case of non-constant k

We retain equation (2.3) but allow k to be a variable scalar subject to the vanishing of its Lie derivative with respect to the Killing vector ξ^μ, i.e.

$$k_{,\mu}\xi^\mu=0\rightarrow k_{,\mu}v^\mu=0. \tag{3.1}$$

Using the definitions (2.1) and (2.2), we find that

$$B^{\mu} = -2k\omega^{\mu}, \tag{3.2}$$

$$E_{\mu} = -k\dot{v}_{\mu} + k_{,\mu}. \tag{3.3}$$

Combining equation (3.3) with (2.14), which is still valid, we obtain

$$\dot{v}_{\mu} = \frac{k_{,\mu}}{k + \rho/\sigma}, \tag{3.4}$$

$$E_{\mu} = k_{,\mu}/(1 + k\sigma/\rho). \tag{3.5}$$

Equations (3.4) and (3.5) show that the three vectors E_{μ}, $\dot{v}_{\mu}$ and $k_{,\mu}$ are in the same direction.

Equation (2.27) on substitution of (3.2) now gives

$$\eta^{\alpha\beta\lambda\mu}(\dot{v}_{\lambda}v_{\beta}B_{\mu} - 2kv_{\lambda}\omega_{\mu;\beta} - 2v_{\lambda}\omega_{\mu}k_{,\beta}) = 0. \tag{3.6}$$

Using equations (3.4), (3.5) and (3.6) in equation (2.34), we obtain

$$2S^{\gamma} = (S^{\gamma}/k^2)[\tfrac{1}{2} - (k\sigma/\rho + 1)]$$

so that if $S^{\gamma} \neq 0$,

$$\sigma/\rho = -(4k^2 + 1)/2k. \tag{3.7}$$

Note that if $k^2 = \frac{1}{4}$, (3.7) gives $|\sigma|/\rho = 2$ in agreement with (2.37). In general with k variable, σ/ρ would also be variable but according to (3.7), $|\sigma|/\rho$ has the highest value 2 corresponding to $k^2 = \frac{1}{4}$. Eliminating $\dot{v}_{\mu}$ and σ/ρ from equations (2.24), (3.4) and (3.7), we obtain for $k^2 \neq \frac{1}{4}$

$$\lambda = L|(4k^2 - 1)/k| \tag{3.8}$$

where L is an arbitrary constant. The appearance of such an arbitrary constant need not cause any surprise; the Killing vector ξ^{μ} is arbitrary up to a constant multiplier and hence the arbitrariness in λ (see equation (2.21)). Thus if ξ^{μ} is suitably normalised, L can be taken to be unity.

We now use (3.7) in (3.5), (2.15) and (2.33) to obtain

$$E_{\mu} = -2k_{,\mu}/(4k^2 - 1), \tag{3.9}$$

$$4\pi\rho = -\frac{2k}{4k^2+1}E^{\mu}{}_{;\mu} - E^2 + \frac{2}{4k^2+1}B^2, \tag{3.10}$$

$$4\pi\rho = -\frac{12k^2}{16k^4+4k^2+1}(E^2 - B^2). \tag{3.11}$$

Equation (3.11) shows that even in this case of variable k the magnetic field dominates over the electric field. Eliminating ρ from equations (3.10) and (3.11), we obtain

$$2(16k^4 + 4k^2 + 1)kE^{\mu}{}_{;\mu} + (4k^2 - 1)^2(4k^2 + 1)E^2 - 2(4k^2 - 1)(2k^2 + 1)B^2 = 0. \tag{3.12}$$

Equation (3.12) is in agreement with equation (2.39) in the case $k^2 = \frac{1}{4}$. Eliminating E^{μ} from equations (3.12) and (3.9) and using the form (2.28) for B^{μ}, we obtain

$$(64k^6 - 1)k^3k^{\mu}{}_{;\mu} - k^2(64k^6 + 48k^4 + 12k^2 - 1)k_{,\mu}k^{,\mu}$$
$$- \tfrac{1}{2}(2k^2 + 1)(4k^2 - 1)^3\psi_{,\mu}\psi^{,\mu} = 0. \tag{3.13}$$

Again substituting (2.28) in (2.17) and keeping in mind that k is now not a constant, we obtain

$$[\psi_{,\mu} k^4/(4k^2-1)^3]^{;\mu} = 0 \qquad (k^2 \neq \tfrac{1}{4}). \tag{3.14}$$

Equations (3.13) and (3.14) may form the basis for obtaining solutions in particular cases. In fact, if axial symmetry is assumed and a Weyl-type coordinate system introduced, the divergences in the above equations will be reduced to ordinary Laplacians in two dimensions and trial solutions (albeit somewhat complicated) may not be difficult to obtain. However in the present paper it is not our purpose to go into that exercise.

The situation is different if S^γ vanishes. Equation (3.7) and all that follow do not apply. In view of equation (2.24) and (3.4) it follows that λ, k and σ/ρ are functionally related. Writing

$$\lambda_{,\mu}/\lambda = g(k)k_{,\mu} \tag{3.15}$$

where $g(k)$ is a function of k, one has

$$\rho/\sigma = (1-kg)/g, \tag{3.16}$$

$$E_\mu = k_{,\mu}(1-kg). \tag{3.17}$$

Equations (2.15) and (2.33) now give

$$4\pi\sigma = (1-kg)k_{,\mu}{}^{;\mu} - k(g'+g^2-f^2/k^2)k_{,\mu}k^{,\mu}, \tag{3.18}$$

$$4\pi\rho\left(1-\frac{g^2}{(1-kg)^2}\right) = \left[(1-kg)^2 - g^2 + 2f^2 + (1-gk)^{-1}\left(g'+g^2-\frac{f^2}{2k^2}-\frac{f^2 g}{2k}\right)\right]k_{,\mu}k^{,\mu} \tag{3.19}$$

where

$$B_\mu = f(k)k_{,\mu}$$

and because of equation (2.17)

$$fk_{,\mu}{}^{;\mu} = -k_{,\mu}k^{,\mu}(f'+2fg-f/k). \tag{3.20}$$

Obviously one can eliminate σ and ρ from equations (3.16), (3.18) and (3.19) to obtain an equation involving $k_{,\mu}{}^{;\mu}$, $k_{,\mu}k^{,\mu}$ and the two scalars f and g and their first-order derivatives with respect to k. This equation along with (3.20) would give two equations for the three unknown functions f, g and k. One has thus the freedom to adjoin a suitable relation between the three functions to simplify the equations. Again it is not our purpose to enter into these details in this paper.

4. Concluding remarks

The main results of the paper may be summed up as follows. If one assumes for a distribution of charged dust in rigid rotation the relation (2.3), then for constant k, one has in general $k^2 = \frac{1}{4}$ and $|\sigma|/\rho = 2$ except when the Poynting vector in the rest frame of the dust vanishes. Kinematically this vanishing corresponds to an alignment of the vorticity and acceleration vectors. In this latter case one can apparently obtain solutions with arbitrary (but constant) values of σ/ρ. When k is not a constant, $|\sigma|/\rho$

has, in general (i.e. if the Poynting vector does not vanish), a simple relation with k and has a maximum possible value of two. With the Poynting vector vanishing in the case of variable k, there is an additional degree of freedom. A notable feature of the present investigation is that no symmetry assumption has been introduced and the results have been obtained in a coordinate-independent manner.

The alignment of the electromagnetic potential and the velocity vectors, which is the crucial assumption of the paper, has not been further investigated. It leads to an alignment of the vorticity vector with the magnetic field. While this may be a mathematically appealing situation, in nature this condition is perhaps rarely realised. In particular, in the case of neutron stars which may be supposed to be seats of intense magnetic fields, the magnetic dipole moment is perhaps inclined to the axis of rotation giving rise to radio emission. Such an energy emission, of course, would make even the stationary assumption invalid.

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